KERR UNIQUENESS AND WHAT TRAPPING 'S GOT TO DO WITH IT

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Abstract. These are lecture notes for an informal lecture at Mittag Leffler. They are accordingly sluggish, brief and incomplete. They should however give a good overview of the problem and the ideas and fractional results floating around.
Warning: the discussion of the literature is not very rigorous and the exact conditions used and statements made might be different. The goal here is just to convey the general ideas.

Contents

1. The Problem

One of the open conjectures in General Relativity is, whether in 3+1 dimensions the Kerr family of spacetimes are the only asymptotically flat, stationary (T from here on being the Killing vector field that is timelike at infinity) black hole solution (there exists an event horizon H^+, H^- with positive surface gravity $\kappa > 0$) to the Einstein vacuum equations ($R_{\mu\nu} = 0$). The first uniqueness theorem for black holes was proven by Birkhoff in spherical symmetry. Since then, there have been numerous results for static spacetimes with varying assumptions.

Speaking of assumptions: I will assume in the following $(\mathcal{M}, g)$ to be a 3+1 dimensional manifold with Lorentzian metric with signature $(-,+,+,+)$. $\mathcal{M}$ is a globally hyperbolic, asymptotically flat black hole spacetime with boundary $H^+ \cup H^-$. (i.e I m only concerned with the domain of outer communications here)

A uniqueness result is known under the additional assumption of axi-symmetry due to Carter and Robinson (these proves must make use of the vacuum EFE right?). Hawking demonstrated that there exists a second Killing vector field on the horizon and then used Cauchy-Kovalevskaya to extend it off the Horizon. Then one can apply the result by Carter and Robinson to show the full result. The Problem here is that Cauchy-Kovalevskaya uses analyticity and hence Hawking's result only proves uniqueness within the class of analytical solutions of the Einstein vacuum equations. This is very restrictive. One result by Klainerman and collaborators shows that Kerr is unique in a neighbourhood of Kerr. I.e. if you assume that you

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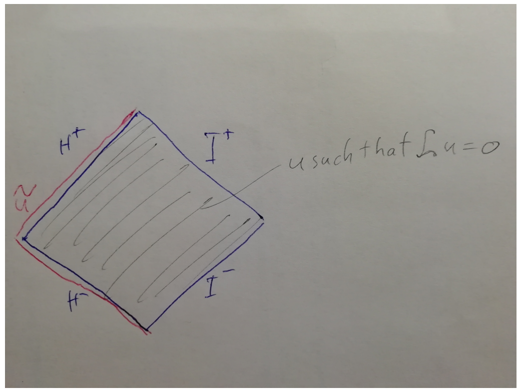

FIGURE 1. Set up for the unique continuation.

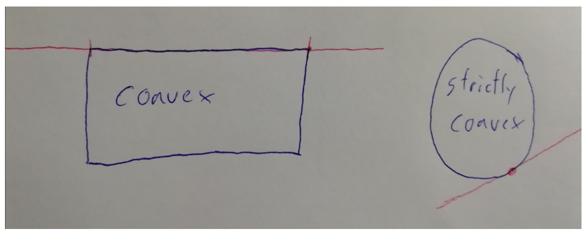

FIGURE 2. Difference between convexity and strict convexity.

have a stationary black hole spacetime that is close to Kerr then you have to be Kerr.

To go beyond analyticity and small data the main obstacle is the ergoregion. I.e. the region around the horizon where T is not time like. I will use the following notation from here on

$$\text{Ergoregion} \qquad \mathcal{ER} := \{x \in \mathcal{M} | g(T,T)|_x \geq 0\} \tag{1.1}$$

$$\text{Ergosphere} \qquad \mathcal{ES} := \{x \in \mathcal{M} | g(T,T)|_x = 0\} \tag{1.2}$$

Not that it is clear that the ergoregion will always be constrained to a bounded region around the event horizon because we assume that $g(T,T) < 0$ in the asymoptotically flat end.

To show uniqueness, beyond analyticity we need to extend the Killing vectorfield of the horizon in whatever regularity class we are interested in (for simplicity let's take smooth solutions). This boils down to proving a unique continuation result for ill-posed data for a hyperbolic equation.

Let $\tilde{u}$ be data on the event horizon $H^+ \cup H^-$ and $\mathcal{L}u = 0$ be our equation of interest. We then want to show that there exists only one u that satisfies the equation and further satisfies $u|_{H^+ \cup H^-} = \tilde{u}$. See Figure 1 for an illustration.

One way to prove the unique continuation is to prove Carleman estimates. This was first achieved by Klainerman et al. using strict pseudo convexity of the bifurcation sphere to extend the Killing Vector field to an ϵ neighbourhood of the horizon. Strict pseudo convexity of a surface means, that every null geodesics tangent to a surface bounces off of it to the same side. See Figure 2 for a illustration of the difference between convexity and strict convexity. This result was recently reproduced by Oliver Lindblad Petersen using the assumption of compactness of the Cauchy horizon. Now this part works all fine for general stationary space times. However if one wants to continue the solution outside a narrow neighbourhood around the

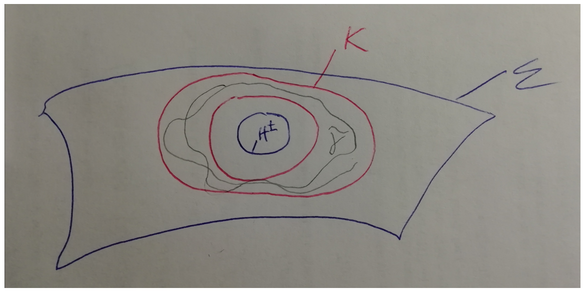

FIGURE 3. Illustration of the definition of trapping $\tilde{\gamma}$ never leaves K.

horizon, things become more difficult. A unique continuation theorem for solutions to a wide range of hyperbolic equations was proven for Kerr space times by Klainerman et al. using a property called T-pseudo convexity. A hyper-surface is strictly T-pseudo convex if the strict convexity holds for all null geodesics γ with $g(\dot{\gamma}, T) = 0$. In fact to prove unique continuation what we need to do is to find a "sufficiently" T-pseudo convex foliation of the domain of outer communication. For Kerr this is given in Boyer Lindquist coordinates by the $r = const$ hypersurfaces. (more on Kerr later)

If one is to prove uniqueness by this strategy one need to show that there exists a "sufficiently" T-pseudo convex foliation of the DOC in every stationary black hole space time. Now what are potential obstacles to the existence of a "sufficiently" (T-) pseudo convex foliation of the DOC? The biggest problem is trapping because the best that can happen with a trapped null geodesic is that you find a foliation in which the null geodesic stays in one leaf. In every other foliation the trapped null geodesic will oscillator between an outermost and an innermost leaf. Thus either violating the condition of convexity or that of strictness. Now we know that trapping is a generic feature of black hole space times so it is clear that a strictly pseudo convex foliation can not exist. But can we show that a strictly T-pseudo convex foliation exists? For that one place to start is to try and rule out the existence of trapped T-orthogonal null geodesics. Now lets write down what we mean by a null geodesic being trapped in a stationary space time. For simplicity I will assume that all orbits of T in the DOC extend from i^- to i^+, hence that they all intersect every Cauchy surface (this is clear where T is timelike and is probably generically true away from the horizon by some continuity arguments for orbits on T. Potentially there is something like that in Mars Reiris. However on the horizon this clearly fails.).

Definition 1 (Trapping)**.** *Let $\gamma(\lambda)$ be a null geodesic and Σ a Cauchy hypersurface in the DOC. Let $\tilde{\gamma}(\lambda)$ in Σ be the path of the intersection $\Phi_\tau(\gamma(\lambda)) \cap \Sigma$ for $\tau \in (-\infty, \infty)$.*

γ is called trapped if $\tilde{\gamma}(\lambda)$ remains in a compact region K away from the horizon.

See Figure 3 for an illustration.

Recall that for γ being a T-orthogonal null geodesic means $g(\dot{\gamma}, \dot{\gamma}) = g(\dot{\gamma}, T) = 0$ and hence from asymptotic flatness and stationarity ($g(T, T) \to -1$ at infinity) it is clear that T- orthogonal null geodesics are confined to a compact region around the event horizon. Hence to prove existence/non-existence of T-orthogonal trapping it reduces to the question whether every T-orthogonal null geodesic comes arbitrarily close to the horizon or not.

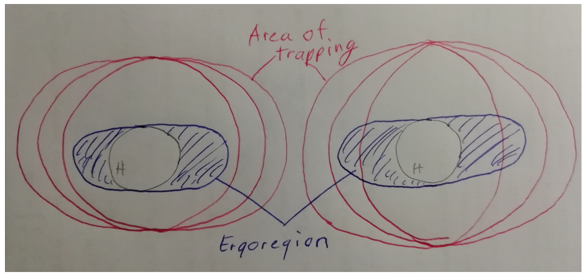

FIGURE 4. The r, θ plane in Boyer Lindquist coordinates in Kerr. Left: the case of $a/M < 1/\sqrt{2}$ the ergo region and trapping do not intersect. Right: in the case of $a/M > 1/\sqrt{2}$ they do. H indicates the horizon.

2. Kerr

Now this is a good point to take a look at Kerr. (for a detailed discussion see lecture notes arxiv 1611.06927). First lets look at trapping and the ergo region in physical space. See Figure 4 for an illustration of the r, θ plane in Boyer Lindquist coordinates. We observe two things: First, for small $a/M < 1/\sqrt{2}$ the Ergoregion and the area of trapping do not intersect and hence there is certainly no T-orthogonal trapping there. For large $a/M > 1/\sqrt{2}$ they do intersect and hence we need a more detailed analysis. Second, the ergoregion has one connected component around the event horizon and the ergosphere touches the horizon at two points (on the rotation axis). $g(T, T)$ satisfies a monotonicity property along the congruence of null geodesics generated by the principle null directions and hence by symmetries in BL coordinates $\partial_r g(T, T) \neq 0$ in the DOC.

Now if we look at the geodesic equations they separate in BL coordinates and we get

$$\dot{\theta}^2 = \Theta(\theta, E, L_z, Q) = L_z^2 \Theta(\theta, E/L_z, 1, Q/L_z^2) \tag{2.1}$$

$$\dot{r}^2 = R(r, E, L_z, Q) = L_z^2 R(r, E/L_z, 1, Q/L_z^2) \tag{2.2}$$

Global behaviour of null geodesics is defined by 2 degrees of freedom. The homogeniety in L_z comes from the freedom of affine reparametrization. Note here $E = -g(\dot{\gamma}, T)$, $L_z = g(\dot{\gamma}, \partial_\phi)$ and Q is the Carter constant coming from a hidden symmetry. So now we can analyse the behaviour of null geodesics from a "phase space" perspective. First lets look at $R(r, E/L_z, 1, Q/L_z^2)$ (a Mathematica file to play with these things is linked in the lecture notes mentioned above). In Figure 5 we plot $R = 0$ in the $E/L_z, r$ plane. We see immediately that the location of trapping in phase space has a gap around $E/L_z = 0$ and hence we do not have T-orthogonal trapping in Kerr (and neither in stationary black hole space times close to Kerr if they were to exist)

Before we go to some new ideas and conjectures that might help to solve the problem, one final observation. Let us define $r_E(Q)$ as the radius such that $R(r_E, 0, Q) = 0$. From the $\dot{\theta}$ equation we get a cone $\theta_{min}(Q) \leq \theta \leq \theta_{max}(Q)$ around the equatorial plane where these T-orthogonal null geodesics can move in Kerr. Now if we intersect this cone with the $r = r_E(Q)$ sphere then unsurprisingly the intersection lies at the point where the cone intersects the ergosphere. See Figure 6 for an illustration.

Before we move on, a quick remark: since T-orthogonal trapping does not exist in Kerr, if one could demonstrate that a stationary black hole spacetime featuring trapped T-orthogonal null geodesics exist, then one would immediately have a counter example to Kerr uniqueness.

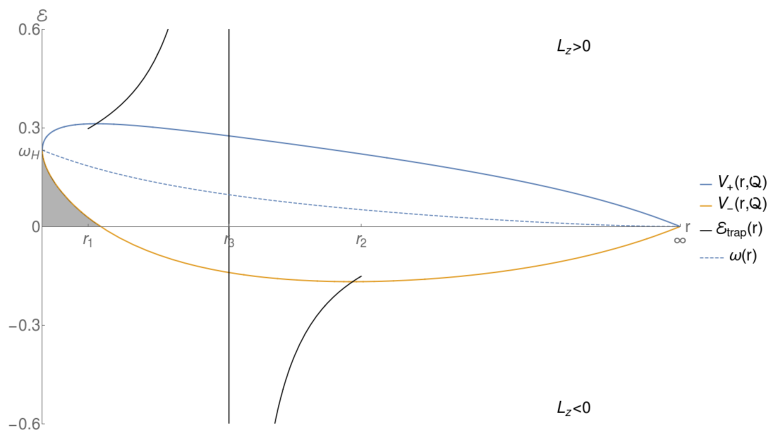

FIGURE 5. $R = 0$ in the $E/L_z, r$ plane for $a = 0.764$ and $\mathcal{Q} = 0.18$. Its qualitative features are preserved when a and $\mathcal{Q}$ are changed. The location of trapping in phase space is indicated by the function $\mathcal{E}_{trap}(r)$. The extrema of the pseudo potentials are the intersection of V_+ and V_- with this function. Therefore they slide on this curve as $\mathcal{Q}$ increases. The area filled in gray corresponds to geodesics with $E < 0$. It is clear from this plot that the regions occupied by geodesics of negative energy and trapped geodesics respectively are disjoint in phase space.

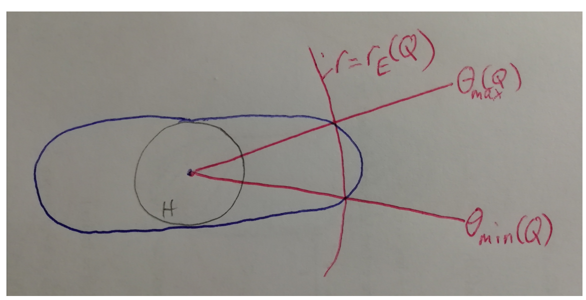

FIGURE 6. The intersection of $r = r_E$ with the θ cone coincides with the intersection of the ergosphere with the θ cone.

3. Brain farts and Conjectures plus some fractional results

So what can we say about the general case? First, let's be clear, this is a pretty hard problem, because there exists very little structure that one can use. So I'll start with the obvious obstructions to a prove that T-orthogonal trapping can not exist, and a direct obstruction to uniqueness itself. The most straight forward one being if the ergoregion could have from the horizon disconnected components. A) this doesn't happen in Kerr and B) this immediately gives T-orthogonal trapping as T-orthogonal null geodesics can exist in the disconnected components of the ergoregion but can never leave it. So for a direct proof one needs to show the following conjecture.

Conjecture 1 (Monotonicity). *$g(T, T)$ satisfies a suitable monotonicity condition.*

Since we can not assume the existence of principle null directions we might need some auxiliary structure. Now we know that both the horizon and $I^{\pm}$ have topology $\mathbf{R} \times S^2$. Let u be a coordinate function on the $\mathbf{R}$ part of H^+ and v be a coordinate function on the $\mathbf{R}$ part of I^+ then something along the following lines might be helpful if it can be defined in a sensible manner (my gut feeling is probably not)

Conjecture 2 (Earliest arrival)**.** *Let L^+ be the future light cone emanating at p. For every point p in the DOC there exists a null geodesic γ_H through p such that*

$$\min[u[L^+ \cap H^+]] = u[\gamma_H \cap H^+] \tag{3.1}$$

and a unique γ_I through p such that

$$\min[u[L^+ \cap I^+]] = u[\gamma_I \cap I^+] \tag{3.2}$$

for every "good" coordinate functions u and v.

These are very generic statements and hence likely hard to prove. So lets have a look at something where I can actually make a partial statement.

Lemma 1. *Let γ be a T-orthogonal null geodesic and $\gamma(0) \in \mathcal{ES}$ then either*

(1) $\exists C > 0$ such that for all $C > \epsilon > 0$ we have that $\gamma(\pm\epsilon) \in \mathcal{ER} \backslash \mathcal{ES}$
(2) $\gamma(\lambda) \in \mathcal{ES}$ for all $\lambda \in (-\infty, \infty)$.

If not 1) then we can use the symmetry to move the piece around and it will stay in the ergosphere hence we are in case 2). To get an interesting partial result one would need to show the following conjecture.

Conjecture 3. *2) implies that $\gamma(\lambda) \in \mathcal{ES} \cap H^+$.*

This conjecture would imply that T-orthogonal null geodesics either bounce of the ergosphere or they lie in the intersection of the ergosphere with the horizon. (This might be enough to show that the ergoregion can't have disconnected components) This seems to be a possible result, given that T is tangent to the ergosphere and hence 2) would imply that the ergosphere is a null surface along γ so one might show some contradiction if it were away from the horizon (using null expansion, Killing horizons and such stuff).

At this point it is worth noting that $\mathcal{ES} \cap H^+ \neq \emptyset$ because $H^\pm$ have topology $\mathbf{R} \times S^2$ and hence if we decompose the generator as $K = \alpha T + \phi$ then we can always choose α such that ϕ lives on S^2 (because T has non compact orbits) and therefore $\phi = 0$ on at least two points on S^2 (is it exactly 2 points if ϕ is Killing on S^2?). There we have $K = \alpha T$ and thus $g(T, T) = 0$. This gives rise to the following questions/conjecture

Conjecture 4. *If we extend ϕ off the horizon, will its orbits still be closed (i.e. does it always lie on S^2) and does $\phi = 0$ form continuous sets in the manifold. Does T become timelike along this set as we go off the horizon? Or more generally do ϕ and T always span a timelike direction away from the horizon? (apparently that is known and true according to comments during the lecture)*

As far as I am aware the proof by Carter and Robinson makes use of the fact that in axisymmetry T and ϕ always span a timelike direction. I guess the argument might carry over to the situation where we know the existence of ϕ apriori only close to the horizon. The strategy would then be to bootstrap your way out of the ergoregion.

3.1. **Indirect Approach.** And now for something completely different. Let (e_0, e_i) be an orthonormal frame at a point p then we can write the tangent vector to every null geodsic at p as

$$\dot{\gamma} = \beta(e_0 + k_i e_i) \tag{3.3}$$

with $|\vec{k}| = 1$. Hence the vectors $\vec{k} \in S^2$ which is usually referred to as the celestial sphere. By fixing β for all k we select a cross section of the lightcone at p. Changing β corresponds to the freedom of affine reparametrization for null geodesics. In

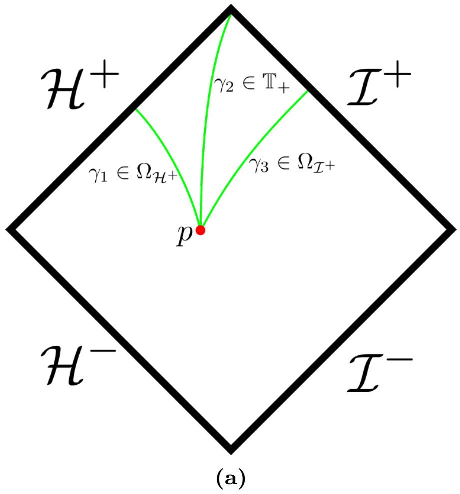

FIGURE 7. Conformal diagrams giving a schematic representation of elements of the sets on the celestial sphere.

the following I will use $\gamma(k|_p)$ to denote the null geodesic through p associated with a direction k on the celestial sphere. Now we can define a couple of sets on S^2 depending on the global properties of the null geodesics associated with these directions.

Definition 2. *The future infalling set:* $\Omega_{\mathcal{H}^+}(p) := \{k \in S^2 | \gamma(k|_p) \cap \mathcal{H}^+ \neq \emptyset\}$.
The future escaping set: $\Omega_{\mathcal{I}^+}(p) := \{k \in S^2 | \gamma(k|_p) \cap \mathcal{I}^+ \neq \emptyset\}$.
The future trapped set: $\mathbb{T}_+(p) := \{k \in S^2 | \gamma(k|_p) \cap (\mathcal{H}^+ \cup \mathcal{I}^+) = \emptyset\}$.
The past infalling set: $\Omega_{\mathcal{H}^-}(p) := \{k \in S^2 | \gamma(k|_p) \cap \mathcal{H}^- \neq \emptyset\}$.
The past escaping set: $\Omega_{\mathcal{I}^-}(p) := \{k \in S^2 | \gamma(k|_p) \cap \mathcal{I}^- \neq \emptyset\}$.
The past trapped set: $\mathbb{T}_-(p) := \{k \in S^2 | \gamma(k|_p) \cap (\mathcal{H}^- \cup \mathcal{I}^-) = \emptyset\}$

It follows immediately from continuous dependence on initial data for null geodesics, that the Ω_* sets are all open and as a consequence $T^{\pm}$ are closed. Now if we assume trapping to be unstable in the sense specified below (to prove instability of trapping from the assumptions in the uniqueness conjecture alone is a whole other can of worms I don't want to get into. There are fractional statements in my thesis and a couple messy notes on my computer for those interested), we can easily prove the following statement:

Lemma 2. *Let $\mathcal{M}$ be a smooth stationary Lorentzian manifold with Killing vector field T with one asymptotically flat end. Further assume future and past trapping to be unstable in the sense that for any observer at any point p in the exterior region we have that for any $k \in \mathbb{T}_+(p)$ for any $\epsilon > 0$*

- $B_\epsilon(k) \cap \Omega_{\mathcal{H}^+}(p) \neq \emptyset$
- $B_\epsilon(k) \cap \Omega_{I^+}(p) \neq \emptyset$.

then any trapped null geodesic γ in the exterior region satisfies $-g(T, \dot\gamma) \geq 0$.

Proof. Note that for every $k \in \Omega_{I^+}(p)$ we have $-g(\dot\gamma(k|_p), T) > 0$. Due to the instability condition we can choose a convergent sequence $q_i \in \Omega_{I^+}(p)$ with $\lim_{i\to\infty} q_i = k$ for any $k \in \mathbb{T}_+(p)$. We then have that

$$E(k) = -g(\dot\gamma(k|_p), T) = \lim_{i\to\infty} -g(\dot\gamma(q_i|_p), T) \geq 0 \tag{3.4}$$

The statement then follows from the fact that the trapped set is given by $\mathbb{T}(p) := \mathbb{T}_+(p) \cap \mathbb{T}_-(p)$. ∎

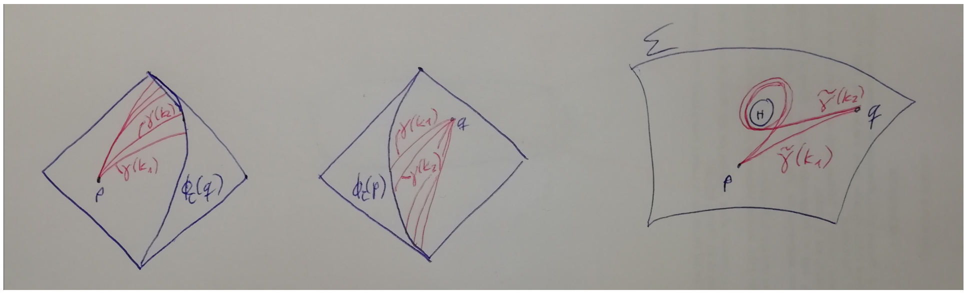

FIGURE 8. Left: there are infinitely many future directed null geodesics from p to $\phi_\tau(q)$. Middle: there are infinitely many past directed null geodesics from q to $\phi_\tau(p)$. Right: in Σ they trace out the same path

Now again the result is not quite strong enough to be really interesting. What one would need to show is the following conjecture.

Conjecture 5. *For every $k \in \Omega_{I^+}(p)$ we have $-g(\dot{\gamma}(k|_p), T) \geq \delta > 0$.*

In the following a couple observations. The reason to believe the conjecture to be true lies in the fact that choosing e_0 fixes a length scale on the light cone. This is essential for there to exist a uniform bound. It is easy to understand at infinity where we have $g(\dot{\gamma}, T) < 0$ from the fact that T is timelike. Choosing $\tilde{e}_0 = T/|T|$ and setting $\beta(k, \tilde{e}_0) = 1$ we get that $g(\dot{\gamma}(k|_p), T) = 1$ for all $k \in S^2(p)$. Combining this with the observation that if we fix the scaling $\beta(k, e_0) = 1$ for any particular e_0 for all $k \in S^2$ then we have that

$$\frac{\sqrt{1-v}}{\sqrt{1+v}} \leq \beta(k, \tilde{e}_0) \leq \frac{\sqrt{1+v}}{\sqrt{1-v}} \tag{3.5}$$

where $v \in [0, 1)$ is the fraction of the light speed of the boost between e_0 and $\tilde{e}_0$. Hence setting $\beta(k, e_0) = 1$ for any e_0 timelike gives $g(\dot{\gamma}(k|_p), T) \geq \delta$ where delta is determined by the boost between e_0 and $\tilde{e}_0$.

Now for the last set of ideas I will assume that for every p, q in $\mathcal{M}$ with q in the asymptotically flat end there exists an infinite set of $k_i \in S^2(p)$ such that $\gamma(k_i|_p) \cap \phi_\tau(q) \neq \emptyset$. Where $\phi_\tau(q)$ denotes the orbit of q under T. (This statement is true for Kerr and rather generic black hole space times due to a result by Perlick et. al. I don't remember how generic the conditions are though. In fact, if true this would settle the question whether all orbits of T in the DOC extend from i^- to i^+).

This assumption immediately implies that there exists a set of past directed null geodesics at q such that $\gamma(k_i|_q) \cap \phi_\tau(p) \neq \emptyset$. Hence we get a set of k_i at q that we can identify with the k_i at p by demanding that $\tilde{\gamma}(k_i|_p) = \tilde{\gamma}(k_i|_q)$ and hence for every k_i there exists a τ_i such that $\gamma(k_i|_p) = \phi_{\tau_i}(\gamma(k_i|q))$ and it is likely to hold that $\sum_{i=N}^{\infty} |\tau_i - \tau_{i+1}| = \infty$ for all N. See Figure 8 for an illustration. Now it is clear however, that $\tilde{\gamma}(\lim_{i\to\infty} k_i|_p) \neq \tilde{\gamma}(\lim_{i\to\infty} k_i|_q)$. Thus the remaining problem here is to show that somehow $g(\dot{\gamma}(\lim_{i\to\infty} k_i|_q), T) \leq -\delta < 0$ implies that $g(\dot{\gamma}(\lim_{i\to\infty} k_i|_p), T) \leq -\delta < 0$.

One idea would be to parallel transport a unit timelike vector at q to p such that $g(\dot{\gamma}, e_0) = 1$ this would generate a sequence of timelike unit vectors at p with $g(e_0^i, \dot{\gamma}(k_i|_q)) = 1$ and the remaining question is whether this sequence of e_0^i stays away from the light cone. This in a sense amounts to a finite redshift argument between p and q.

One last idea would be to prove that $\gamma(\lim_{i\to\infty} k_i|_p)$ and $\gamma(\lim_{i\to\infty} k_i|_q)$ asymptote to the same trapped null geodesics. Essentially the idea is that the path the $\tilde{\gamma}(k_i|_{p,q})$ take between p and q is highly restricted. So the conjecture is the following.

Conjecture 6. *For every $\epsilon > 0$ there exists an N such that for all $i > N$ we have that $\tilde{\gamma}(k_i)|_q^p \in U_\epsilon^N(p,q) \in \Sigma$ where $U_\epsilon^N(p,q) := \bigcup_{\gamma(k_i)|_q^p} B_\epsilon(\gamma(k_i)|_q^p)$.*

Here with $\tilde{\gamma}(k_i)|_q^p$ I mean the segment of $\tilde{\gamma}$ between p and q. Accordingly $U_\epsilon^N(p,q)$ is a small neighbourhood in Σ connecting p and q. This is basically saying that the region in Σ where the null geodesics spend most of the time, gets tighter and tighter constrained. This might be sufficient to show that the two limiting null geodesics in fact asymptote to the same trapped null geodesic. (the idea is that they spend an increasing amount of time close to a particular trapped nullgeodesic. When they arrive close, then they are close in the tangent bundle because they will stay close for a long time. And hence since one is uniformly constrained away from $g(\dot{\gamma}, T) = 0$ and we can bring them arbitrarily close in the bundle, we can get it close enough that the trapped one can not have $g(\dot{\gamma}, T) = 0$. This argument would require an number of extra steps, as one would need to show that for every p for every $k \in T^+(p)$ there exists a q in the asymptotically flat end such that the sequence $k_i(q)|p$ constructed in this manner in fact converges to k.